Analysis of a Two-Element Array of 1-Dimensional Antennas

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***Abstract*—Radiation, reception and scattering by 1-dimensional antennas was described recently. Arrays of two 1-dimensional antennas with various excitations and loads are analyzed both as radiating antennas and receiving antennas. Antenna parameters computed include: mutual coupling impedance matrix, mutual coupling scattering matrix, element efficiency, element radiation patterns and directivities, array radiation patterns and directivities, element receiver cross-section and received power, scattering and scattered power, complete (local and radiation ports) scattering matrix.**

***Index Terms*—Antenna Arrays, Antenna Theory, Receiving Antennas, Transmitting Antennas, Antenna Scattering.**

I. INTRODUCTION

Generally, antennas are considered to radiate spherical waves as well as to receive and scatter in a 3-dimensional space. Nevertheless, there are special circumstances such as, for example, antennas radiating cylindrical waves and receiving and scattering in the 2-dimensional space between parallel plates. Recently, examples of particularly simple antennas radiating, receiving and scattering plane waves in a 1-dimensional space were described in detail [1]. Such structures are faithfully modeled by transmission line equivalent circuits. Aspects of such arrays were summarized in [2]. Here, we analyze the properties of the particularly simple two-element array of 1-dimensional antennas in greater detail. In view of the relative mathematical simplicity (only elementary functions are required) the analysis of an array one dimensional antennas can serve an important tutorial function.

Consider a uniform plane wave normally incident on an array of two parallel thin resistive sheets separated by a distance ℓ. This structure is faithfully modeled by the transmission line circuit Fig. 1. In this circuit the transmission line models 1-dimensional space, and the two shunt-T circuits represent the receiving antennas, which are terminated in resistive loads that model the resistive sheets. The ideal transformers match input impedance of the load ports so that the antennas meet all requirements for canonical minimum-scattering (CMS) antennas [3]. The wave incident from the left is represented by a wave parameter a_1, defined so that $|a_1|^2$ has dimensions of watts/(meter)^2.

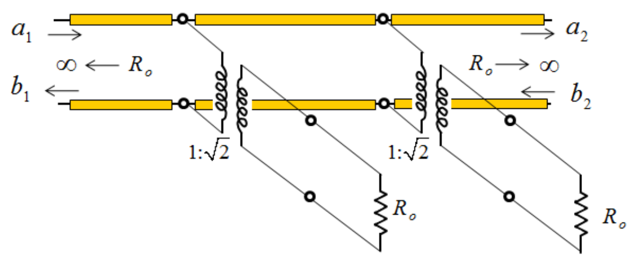

Fig. 1. Two-element array.

II. IMPEDANCE AND SCATTERING MATRICES

The open-circuit impedance matrix for the two-element antenna array illustrated in Fig. 2 is found utilizing the well-known transmission line solutions for voltage and current [4],

$$V(z) = V(z')\cos k(z-z') - jR_0 I(z')\sin k(z-z'), \tag{1a}$$

$$I(z) = -jG_0 V(z')\sin k(z-z') + I(z')\cos k(z-z'). \tag{1b}$$

Using these equations, for example, if the voltage $V(0+)$ and current $I(0+)$ on the transmission line just to the right of the first antenna located at $z'=0$ are known, the voltage $V(\ell-)$ and current $I(\ell-)$ just to the left of the second antenna located at $z=\ell$ are obtained directly.

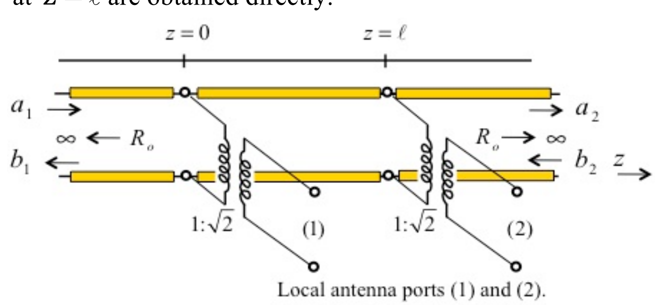

Fig. 2. Transmission line coordinates for open-circuit impedance matrix computation.

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We normalize the characteristic resistance/conductance of the lossless transmission line to unity. The elements of the open-circuit matrix then follow as

$$V(z_1=0)=I_1\sqrt{2}\frac{1}{2},\ I_2=0 \tag{2a}$$

$$Z_{11}=\frac{V_1}{I_1}=\frac{V(z_1=0)\sqrt{2}}{I_1}=1=Z_{22} \tag{2b}$$

$$V(z_2=\ell)=\frac{I_1}{\sqrt{2}}\cos k\ell-j\frac{I_1}{\sqrt{2}}\sin k\ell \tag{2c}$$

$$=\frac{I_1}{\sqrt{2}}\exp\{-jk\ell\}=\frac{I_1}{\sqrt{2}}T \tag{2d}$$

$$Z_{12}=\frac{V(z_2=\ell)\sqrt{2}}{I_1}=\frac{I_1 T}{I_1}=T=Z_{21} \tag{2e}$$

$$Z=\begin{bmatrix} 1 & \exp\{-jk\ell\} \\ \exp\{-jk\ell\} & 1 \end{bmatrix}. \tag{2f}$$

The corresponding normalized voltage scattering matrix for the array is given by $S=(Z-1)/(Z+1)=1-2(Z+1)^{-1}$ where,

$$S_{11}=S_{22}=1-4[(4-\exp\{-j2k\ell\}]^{-1} \tag{3a}$$

$$S_{12}=S_{21}=2\exp\{-jk\ell\}[4-\exp\{-j2k\ell\}]^{-1}. \tag{3b}$$

III. Element Efficiency

The element efficiency η of either element [5] is therefore

$$1-\eta=\left|S_{11}\right|^2+\left|S_{12}\right|^2. \tag{4}$$

This efficiency as a function of the separation or spacing of the two antenna elements is shown in Fig. 3.

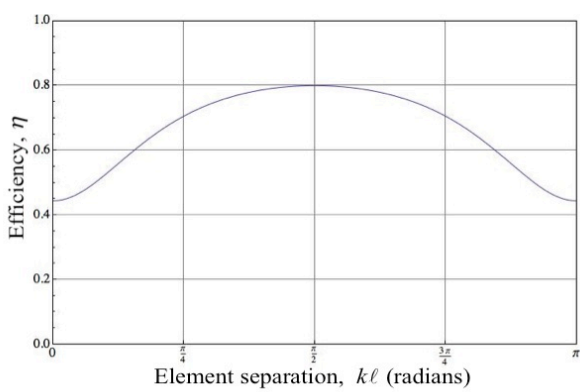

Fig. 3. Element efficiency as a function of antenna array element spacing.

IV. Directivity

Antenna element currents in a terminated array environment for arbitrary incident wave excitation

$$\underline{a}^T=\begin{bmatrix} a_1 & a_2 \end{bmatrix}. \tag{5}$$

are given by

$$\underline{I}=\underline{a}-\underline{b}=[1-\underset{\sim}{S}]a\,. \tag{6}$$

If only the first element is excited $a_1=1.0,\ a_2=0$, the amplitudes towards the far left and the far right may be computed from the currents forced by ideal current sources in the open-circuited array environment

$$b_1(z=0^-)=\sqrt{2}\left[\frac{2-T^2}{4-T^2}\right]=-I_1(z=0^-)\,. \tag{7a}$$

$$a_2(z=\ell^+)=\sqrt{2}\left[\frac{T}{4-T^2}\right]=I_2(z=\ell^+)\,. \tag{7b}$$

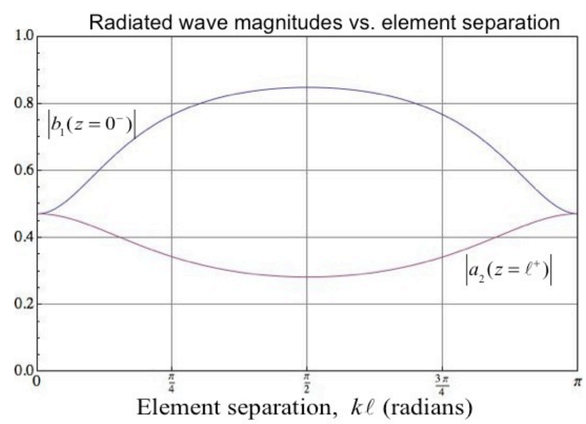

Fig. 4. Far field transmission line wave amplitudes as a function of antenna element separation/spacing.

The directivity of the array is, in general,

$$D(1)=\frac{\left|b_1(z=0^-)\right|^2}{\frac{1}{2}\left[\left|b_1(z=0^-)\right|^2+\left|a_2(z=\ell^+)\right|^2\right]}, \tag{8a}$$

$$D(2)=\frac{\left|a_2(z=\ell^+)\right|^2}{\frac{1}{2}\left[\left|b_1(z=0^-)\right|^2+\left|a_2(z=\ell^+)\right|^2\right]}\,. \tag{8b}$$

and, in particular, the directivity when only the first element is excited is

$$D(1)=\frac{\left|2-T^2\right|^2}{\frac{1}{2}\left[\left|2-T^2\right|^2+1\right]}, \tag{9a}$$

$$D(2)=\frac{1}{\frac{1}{2}\left[\left|2-T^2\right|^2+1\right]}. \tag{9b}$$

These directivities are shown in Figures 5a and 5b.

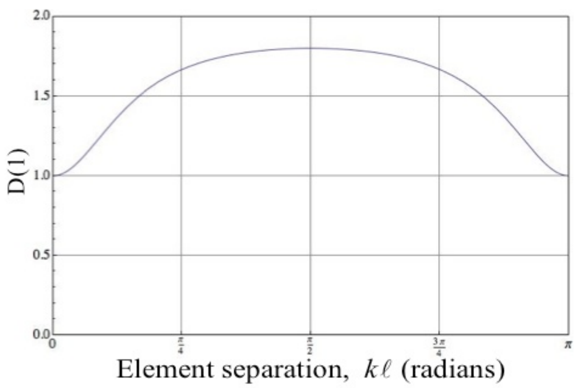

Fig. 5a. Directivity to the right, region 1, in the transmission line when only element #1 of the array is excited in the terminated array environment as a function of antenna element separation/spacing.

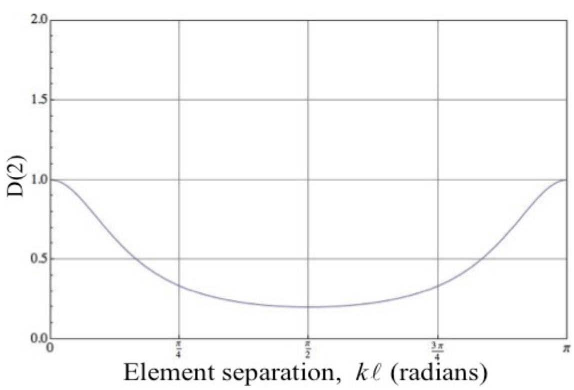

Fig. 5b. Directivity to the left, region 2, in the transmission line when only element #1 of the array is excited in the terminated array environment as a function of antenna element separation/spacing.

In view of the symmetry of the two-element array, if only the second element, element #2, is excited, the above two directivities are simply interchanged.

We note a special feature of directivity in the case of 1-dimensional antennas. There evidently exists an absolute maximum for the directivity, a directivity of 2.0, which is attained when all radiation is concentrated in just one of the two possible directions.

V. Received Power

When a wave is incident on the terminated array from the left, region 1, i.e., $a_1=1$, the power received in the load on the first antenna while the second terminated antenna acts as a passive parasitic director may be found in a number of ways, utilizing different theoretical principles and different points of view.

The received power follows from the previously calculated wave amplitude radiated into the left region 1, eq. (6a), and the principle of network reciprocity. Changing the position of the generator from excitation of the local port #1 to excite the incident wave $a_1=1$ in the left transmission line, has the effect of producing the current $-I_1(z=0^-)$ into the load at the local port #1. The power received in the load at port #1 is theorfore

$$P_{r1}=\left|I_1(z=0^-)\right|^2\cdot R,\quad R=1. \tag{10}$$

The same received power may be calculated by direct circuit calculation of a voltage across the load representing the receiver at port #1. The voltage at the ideal T-junction connecting the first antenna to the transmission line is given by

$$V_1(z=0^-)=[1+S_{11}^R]a_1(z=0-). \tag{11a}$$

Referring this voltage through the transformer, the received power is

$$P_{r1}=\left|\sqrt{2}V_1(z=0^-)\right|^2=2\left|[1+S_{11}^R]a_1(z=0^-)\right|^2 \tag{11b}$$

The scattering matrix reflection coefficient S_{11}^R in the transmission line is computed subsequently in this paper.

Finally, and of particular interest in the present context, the received power may be calculated using antenna theory and the antenna parameters: receiving cross-section $\mathcal{A}$ and directivity.

$$P_{r1}=\mathcal{A}_1(1)\left|a_1(z=0^-)\right|^2 \tag{12a}$$

$$=\bar{\mathcal{A}}\,G(1)[1-\left|S_{11}\right|^2]\left|a_1(z=0^-)\right|^2 \tag{12b}$$

$$=\frac{1}{2}D(1)\eta\left|a_1(z=0^-)\right|^2 \tag{12c}$$

The constant $\bar{\mathcal{A}}$ appearing in eq. (12b) is the universal constant of proportionality relating the gain of a matched antenna system to the corresponding receiving cross-section. This constant has the value $\bar{\mathcal{A}}$ =1/2 for a 1-dimensional antenna, $\bar{\mathcal{A}}$ = λ/2π for a 2-dimensional antenna, and $\bar{\mathcal{A}}$ = $\lambda^2/4\pi$ for 3-dimensional antennas [1].

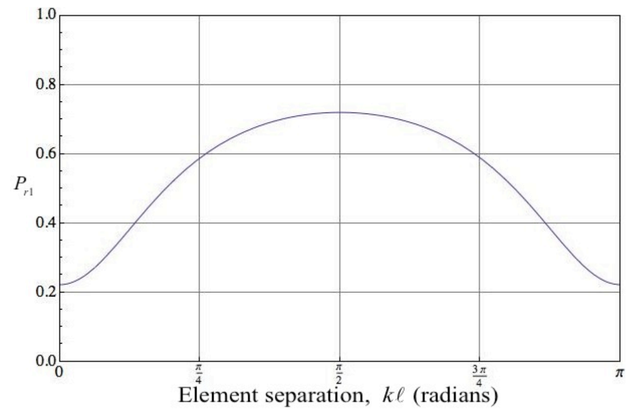

Fig. 6a. Power received by antenna #1, $\left|a_1(z=0^-)\right|=1$, as a function of antenna separation/element spacing.

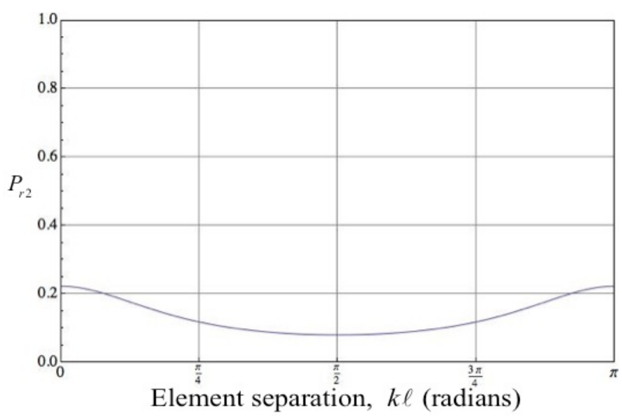

Fig. 6b. Power received by antenna #2, $\left|a_1(z=0^-)\right|=1$, as a function of antenna separation/element spacing.

VI. Scattered Fields and Scattered Power

The scattered field is the difference between the incident field (quantities with 0 superscripts) and the actual field in the presence of the antenna [1]. As a definite measure of scattering, the electromagnetic disturbance of the incident fields produced by the antenna, a quantity with the dimensions of power, termed the scattered power, is often calculated. It should be noted that no general conservation of power relation is associated with the scattered power. In the special case of a canonical minimum-scattering antenna only, the scattered power necessarily equals the received power [1,3].

The incident field is simply the field present in space (the transmission line, the left region 1 and the right region 2) in the absence of scattering structures (presently the two element array antenna). Considering the incident field in space regions 1 and 2 well defined without reference to a scattering structure, it is essential to assure that removal of the scattering structure is carried out in such manner as to reproduce the previously defined space regions and their associated field.

These considerations may be rendered concrete by reference to Figs. 7a and 7b. Absence/removal of the scattering structure, the antenna array, must be implemented in such manner as to preserve phase relations in region 2 that existed in space in the absence of the array. In the case of the present two-element array this is evidently accomplished by removal of the shunt resistors representing the elements and their receiver loads but retaining that part of the array, which is the space of length ℓ separating the elements.

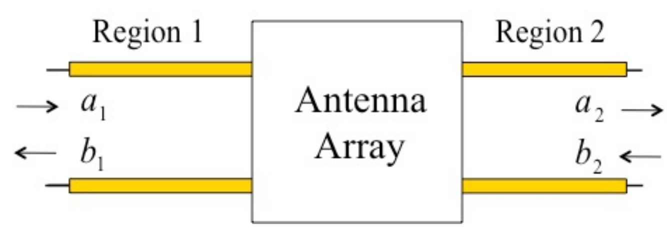

Fig.7a. 1-dimensional space (transmission line) with antenna scattering structure.

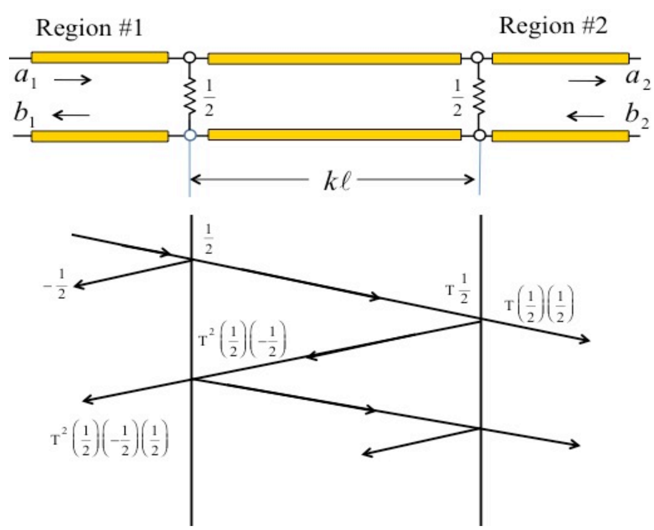

Fig. 7b. Two-element antenna array scattering structure – equivalent transmission line circuit and bounce diagram.

We next compute the scattering matrix S^R controlling the scattering of radiation fields by the array in the 1-dimensional space. The terminal ports of this scattering matrix coincide with the resistors in Fig. 7b, including the resistors themselves within the structure represented by the matrix.

We calculate the desired scattering matrix by means of the bounce diagram shown in Fig. 7b. This makes use of the properties of an individual shunt resistor, Fig. 8.

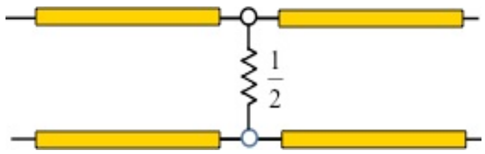

Fig. 8. Individual resistor placed in shunt across a transmission line.

It is easy to derive the scattering matrix for an individual shunt resistor as a function of input reflection coefficient Γ is

$$S_{shr} = \begin{bmatrix} \Gamma & 1+\Gamma \\ 1+\Gamma & \Gamma \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{bmatrix}, \qquad (13)$$

where the matrix is also shown evaluated for Γ corresponding to a normalized shunt resistance ½ seen against a matched load. Following the bounce diagram and summing the resultant geometric series, we readily find the scattering of the match terminated two-element array

$$\underset{\sim}{S}^R = \begin{bmatrix} S_{11}^R & S_{12}^R \\ S_{21}^R & S_{22}^R \end{bmatrix}; \qquad (14a)$$

$$\begin{aligned} S_{11}^R &= -\frac{1}{2} - \frac{T^2/8}{1-T^2/4} = S_{22}^R, \\ S_{21}^R &= -\frac{T/4}{1-T^2/4} = S_{12}^R, \qquad (14b) \\ T &= exp\{-jk\ell\}. \end{aligned}$$

Employing the values just found for the scattering matrix of the terminated array in 1-dimensional space (transmission line) we find in the left region 1:

$$\begin{aligned} a_1^0 &= a_1 \qquad a_1 = a_1 \\ b_1^0 &= 0 \qquad b_1 = \left(-\frac{1}{2} - \frac{T^2/8}{1-T^2/4}\right)a_1, \end{aligned} \qquad (15)$$

and to the right region 2:

$$\begin{aligned} a_2^0 &= Ta_1 \qquad a_2 = \frac{T/4}{1-T^2/4}a_1 \\ b_2^0 &= 0 \qquad b_2 = 0, \end{aligned} \qquad (16)$$

where $T = \exp\{-jk\ell\}$.

The scattered power measure of scattering is

$$P_s = |b_1|^2 + |a_2 - a_2^0|^2. \qquad (17)$$

The above characteristics are concisely summarized in equations (18) – (20), where we also set $a_1 = 1$. The scattered power in the left region 1 $P_s(1)$, the right region 2 $P_s(2)$ and the total scattered power $P_s(total)$ are shown in Fig. 9.

Total fields = Incident fields + Scattered fields

$$\text{Region \#1} \ \{a_1, b_1\} = \{a_1^0, b_1^0\} + \{a_1^S, b_1^S\} \qquad (18a)$$

$$\{1, S_{11}^R\} = \{1, 0\} + \{a_1^S, b_1^S\} \qquad (18b)$$

$$\{0, S_{11}^R\} = \{a_1^S, b_1^S\} \qquad (18c)$$

$$\text{Region \#2} \ \{a_2, b_2\} = \{a_2^0, b_2^0\} + \{a_2^S, b_2^S\} \qquad (19a)$$

$$\{S_{21}^R, 0\} = \{T, 0\} + \{a_2^S, b_2^S\} \qquad (19b)$$

$$\{S_{21}^R - T, 0\} = \{a_2^S, b_2^S\} \qquad (19c)$$

$$\text{Therefore, Scattered Power} = |b_1^S|^2 + |a_2^S|^2 \qquad (20a)$$

$$\text{Scattered Power} = |S_{11}^R|^2 + |S_{21}^R - T|^2 \qquad (20b)$$

$$\text{Scattered Power} = P_s(1) + P_s(2) \qquad (20c)$$

The separate values of scattered power are particularly informative and physically satisfying. As the element spacing approaches a quarter wave $k\ell = \frac{\pi}{2}$ the array absorbs the maximum real power, see Figs. 6a and 6b. Correspondingly, the scattered power in region 1 is a minimum denoting a minimum of back-scatter. On the other hand, in region 2 the scattered power is a maximum, the scattered field deepening the shadow cast by the array through destructive interference. This satisfying physical picture is lost when only the total scattered power is considered.

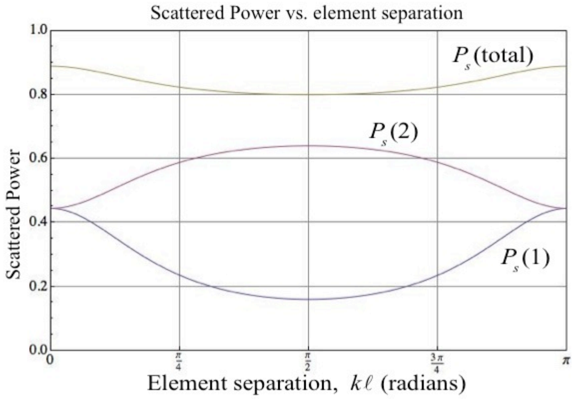

Fig. 9. Scattered power in region 1, region 2 and the total scattered power as a function of antenna array element spacing/separation.

VII. Complete Scattering Matrix Representation of the Antenna in 1-Dimensional Space

A complete scattering matrix representation of the 2-element array antenna, including the two local antenna ports and the radiation (transmission line) ports has the form, in the conventional scattering notation

$$\begin{bmatrix} b_1 \\ b_2 \\ \hline b_3 \\ b_4 \end{bmatrix} = \left[\begin{array}{cc|cc} S_{11} & S_{21} & S_{31} & S_{41} \\ S_{21} & S_{11} & S_{41} & S_{31} \\ \hline S_{31} & S_{41} & S_{33} & S_{43} \\ S_{41} & S_{31} & S_{43} & S_{33} \end{array}\right] \begin{bmatrix} a_1 \\ a_2 \\ \hline a_3 \\ a_4 \end{bmatrix}, \tag{21}$$

wherein reciprocity and the physical symmetry of the array have been explicitly recognized. The first and second rows represent the local ports; the second and third rows represent radiation (transmission line) ports. In the more specialized notation of this paper, transmission line conventions were used to label incident and reflected wave amplitudes in the right hand region 2-transmission line. Consequently we have the following correspondence with the notation of eq. (21)

$$\begin{aligned} b_3 &= b_1(z=0^-) \qquad a_3 = a_1(z=0^-) \\ b_4 &= a_2(z=\ell^+) \qquad a_4 = b_2(z=\ell^+) \end{aligned} \tag{22}$$

The upper left four-element sub-matrix is the local network scattering matrix; the upper right and lower right four-element sub-matrices contain the receive and transmit antenna patterns, respectively; the lower right four element sub-matrix is the spatial (within the transmission line) scattering matrix. See eqs. 23. Since the antenna elements (which excludes the receiver loads) and the 1-dimensional space (transmission line) are lossless, this total scattering matrix is necessarily unitary.

$$S_{11} = 1 - 4\left[4 - T^2\right]^{-1} \tag{23a}$$

$$S_{12} = 2T\left[4 - T^2\right]^{-1} \tag{23b}$$

$$S_{31} = \sqrt{2}\left[2 - T^2\right]\left[4 - T^2\right]^{-1} \tag{23c}$$

$$S_{41} = \sqrt{2}\, T\left[4 - T^2\right]^{-1} \tag{23d}$$

$$S_{33} = S_{11}^R = \left[-2 + \frac{T^2}{2}\right]\left[4 - T^2\right]^{-1} \tag{23e}$$

$$S_{43} = S_{12}^R = T\left[4 - T^2\right]^{-1} \tag{23f}$$

VIII. Conclusions

The evaluation of various parameters that characterize the performance of an array antenna has been exhibited in the simplest context: an array comprising only two identical elements radiating into a 1-dimensional space.

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